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The inverse Jacobi problem

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Abstract. The paper presents a discussion of the relation between the dynamics of a mechanical system based upon a Lagrangian admitting energy conservation and the dynamics based upon its Jacobi Lagrangian, which determines the space trajectories of the system. The basic result found in the paper is the general solution of the inverse problem, i.e. how to determine the full Lagrangian, when as a starting point, an arbitrary homogeneous Lagrangian, which is used to determine the space trajectories of a system, and an arbitrarily assigned energy function which specifies the interaction of the system are given.

1. Introduction

As is known, there are two different variational principles that lead to the equations of motion for test particles in general relativity. If, in a coordinate system $\{x^i\}$, a world line in a spacetime manifold $(\mathcal{M}, g)$ is described by equations of the form $x^i = \xi^i(\tau)$ and $\dot{\xi}^i$ denote the derivatives of the functions ξ^i with respect to the parameter τ, then the first of these principles is based on the action

$$S = \int_{\tau_1}^{\tau_2} \sqrt{g_{ij}\dot{\xi}^i\dot{\xi}^j}\,\mathrm{d}\tau \tag{1}$$

while the action of the second principle reads as

$$W = \tfrac{1}{2}\int_{t_1}^{t_2} g_{ij}\dot{\xi}^i\dot{\xi}^j\,\mathrm{d}t \tag{2}$$

where the parameter is denoted by t, in order to stress the difference between the two dynamics.

Although the Euler–Lagrange equations in both cases are seemingly the same, there is an essential difference between the two actions. In the first case, the action is invariant under arbitrary reparametrizations of the world lines and, as a consequence of the second Noether theorem, only three of the corresponding Euler–Lagrange equations are independent. This indicates that the variational principle based on action (1) determines a world line understood as a locus of points in the spacetime $\mathcal{M}$. The parametrization of the world line is left undetermined in this case. In the second case, the action is invariant only if a constant is added to the parameter and thus action (2) leads to four independent Euler–Lagrange equations which admit, due to the first Noether theorem, a first integral of the energy type. As a result, the variational principle that starts from action (2) determines a world line in $\mathcal{M}$ together with a definite parametrization along it. In other words, it determines a locus

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of points in the space $\mathcal{M} \times \mathbb{R}$, where $\mathbb{R}$ is the real line from which the parameter t in (2) takes its values. However, if one compares the situation with that in the Newtonian dynamics, the product space $\mathcal{M} \times \mathbb{R}$ is not now universal, i.e. it is not now independent of the solution under consideration, and every solution to the dynamical problem determines its own 'time' axis $\mathbb{R}$.

Being prompted by the two examples discussed above, a more general question arises of a possible relationship between a given action of the form

$$W[t_1, t_2; q^i] = \int_{t_2}^{t_1} \mathcal{L}(q^i(t), \dot{q}^j(t))\,\mathrm{d}t \tag{3}$$

where $q^i = q^i(t)$ describes a motion in a configuration space $\mathbb{Q}^n$, i.e. a 'world line' in the space $\mathbb{Q}^n \times \mathbb{R}$, and an action which would describe a trajectory of this motion in the configuration space $\mathbb{Q}^n$ only.

This question is partially answered by the well known Jacobi variational principle [1]† corresponding to an action of the type (3) in which one replaces the description of the motion $q^i = q^i(t)$ in the space $\mathbb{Q}^n \times \mathbb{R}$ by a description of the trajectory $q^i = q^i(\tau)$ in the space $\mathbb{Q}^n$ in terms of a geometric parameter τ that is defined by a procedure based on the energy conservation principle. Jacobi himself complained (in a book by Arnold (1978)) that, in almost all text books, the Jacobi principle had been presented in a rather incomprehensible way. We sympathize with Arnold's conviction that this tradition is also observed by almost all contemporary writers (perhaps even by Arnold himself), and in section 2 we attempt to present a new derivation of the Jacobi principle which, in our opinion, clarifies the issue. This derivation manifestly uses the fact that the Jacobi principle leads to a set of equations that, on the other hand, could have been obtained from the dynamics determined by (3) as a consequence of the elimination of one of the degrees of freedom by solving one of the dynamical equations. The point here is that, in general, in other dynamical theories, in e.g. classical field theory, it is not at all certain whether analogous reduction of some of the degrees of freedom must lead to theories based again on variational principles [3, 4].

In our opinion, in accordance with the just presented point of view, the Jacobi variational principle is a consequence of the following theorem.

Theorem (Jacobi). Let $q^i = q^i(t)$ be a solution of the Euler–Lagrange equations corresponding to action (3). Furthermore, let $x^i = x^i(\tau)$ be a world line in $\mathbb{Q}^n$, being a projection of the motion $q^i = q^i(t)$ from $\mathbb{Q}^n \times \mathbb{R}$ to $\mathbb{Q}^n$. Then there exists a Lagrangian $L_J = L_J(x^i, y^j)$ (where $y^j = \mathrm{d}x^j/\mathrm{d}\tau$) and a corresponding Jacobi action

$$S[\tau_1, \tau_2; q^i] = \int_{\tau_2}^{\tau_1} L_J(x^i(\tau), y^j(\tau))\,\mathrm{d}\tau \tag{4}$$

such that the trajectories are solutions of the Euler–Lagrange equations corresponding to (4). Moreover, the function L_J is homogeneous of degree one in the variables y^j.

The construction of the Jacobi Lagrangian is presented in section 2.

From the Jacobi theorem, it follows that the dynamics based on the Jacobi action principle is reduced to the problem of finding geodesics in a particular Finsler space [5–7]. As a consequence, a natural question arises as to whether the geodesic problem in

† In [1], the Jacobi principle is, not quite correctly, called the Maupertuis principle.

any given Finsler geometry can be considered as being equivalent to the Jacobi dynamics corresponding to a dynamics of the type (3) determined by a certain Lagrangian $\mathcal{L}$. Here, this second problem is called the inverse Jacobi problem and its solution is presented in section 3. In section 4 two simple examples are worked out which illustrate the formalism developed in the previous sections. Another application and some basic prerequisites to the problem are presented in [8].

Returning to the two examples quoted at the beginning of the paper, we see that the action (1) is the Jacobi action corresponding to the dynamics determined by (2).

2. The Jacobi action principle

As is well known, the Euler–Lagrange equations of a dynamical system, described by the action (3), are equivalent to the Hamilton action principle which requires that the variation $\delta W = 0$ for equal time variations of the dynamical variables q^i such that $\delta q^i(t_1) = \delta q^i(t_2) = 0$. As the first step towards a derivation of the Jacobi principle, let us reformulate the Hamilton action principle by replacing the time variable t by an arbitrary parameter τ. Let us assume that the change in parametrization is determined by a smooth and monotonic function θ in the form $t = \theta(\tau)$. When varying the action with respect to the reparametrized world lines, one must take into account the fact that the change in parametrization may depend on the world line. This is done by assuming that θ is an additional dynamical variable. Thus, the new dynamical variables of the problem are now (x^i, θ), $i = 1, \ldots, n$, where $x^i(\tau) = q^i(\theta(\tau))$ and the action is

$$W[\tau_1, \tau_2; x^i, \theta] = \int_{\tau_1}^{\tau_2} \Lambda(x^i(\tau), y^j(\tau), \dot{\theta}(\tau))\, \mathrm{d}\tau \tag{5}$$

where $y^j(\tau) = (\mathrm{d}x^j/\mathrm{d}\tau)(\tau)$, $\dot{\theta}(\tau) = (\mathrm{d}\theta/\mathrm{d}\tau)(\tau)$ and

$$\Lambda(x^i, y^j, \dot{\theta}) = \mathcal{L}\left(x^i, \frac{y^j}{\dot{\theta}}\right)\dot{\theta}. \tag{6}$$

The new Lagrange function Λ is a homogeneous function of degree one in the variables $(y^j, \dot{\theta})$, and therefore the system still has n degrees of freedom even if it is described by $n+1$ dynamical variables. The equal time variations will now correspond to variations with, in general, different values of τ on the world lines, say $\tilde{\tau}$ and τ, respectively, and therefore in the reformulated Hamilton principle one must use complete variations of the new dynamical variables.

Thus, two kinds of variation will be needed: the complete variations

$$\bar{\delta}\theta(\tau) = \tilde{\theta}(\tilde{\tau}) - \theta(\tau) \qquad \bar{\delta}x^i(\tau) = \tilde{x}^i(\tilde{\tau}) - x^i(\tau) \tag{7}$$

and the variations without the variation of the parameter

$$\delta\theta(\tau) = \tilde{\theta}(\tau) - \theta(\tau) \qquad \delta x^i(\tau) = \tilde{x}^i(\tau) - x^i(\tau) \tag{8}$$

where the tilde sign denotes that the corresponding variable is taken along a varied world line. As a result, the obvious relationships must be observed

$$\bar{\delta}\theta(\tau) = \delta\theta(\tau) + \dot{\theta}(\tau)\delta\tau(\tau) \qquad \bar{\delta}x^i(\tau) = \delta x^i(\tau) + y^i(\tau)\delta\tau(\tau) \tag{9}$$

where $\delta\tau = \tilde{\tau} - \tau$ is the variation of the parameter. Furthermore,

$$\delta y^j = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^j\right) \qquad \text{and} \qquad \bar{\delta} y^j = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^j\right) + \dot{y}^j \delta\tau \tag{10}$$

where

$$\bar{\delta} y^j(\tau) = \frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\tilde{\tau}}(\tilde{\tau}) - \frac{\mathrm{d}x^j}{\mathrm{d}\tau}(\tau) \qquad \delta y^j(\tau) = \frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\tau}(\tau) - \frac{\mathrm{d}x^j}{\mathrm{d}\tau}(\tau).$$

The fact that there are equal time variations in the original Hamilton principle now imposes the condition

$$\bar{\delta}\theta(\tau) = \tilde{\theta}(\tilde{\tau}) - \theta(\tau) = 0 \tag{11}$$

which, by virtue of (9), means that

$$\delta\theta(\tau) = -\dot{\theta}(\tau)\delta\tau(\tau) \tag{12}$$

i.e. only one of the variations $\delta\theta$ and $\delta\tau$ is independent. Moreover, for the complete variation we have

$$\bar{\delta} x^i(\tau) = \tilde{q}^i(\tilde{\theta}(\tilde{\tau})) - q^i(\theta(\tau)) = \delta q^i(t) \tag{13}$$

which means that it is equal to the equal time variation of the old dynamical variable q^i.

Taking into account relations (6)–(13), one obtains that the complete variation of action (5) is

$$\begin{aligned}
\bar{\delta} W &= W[\tilde{\tau}_1, \tilde{\tau}_2; \tilde{x}^i(\tilde{\tau}), \tilde{\theta}(\tilde{\tau})] - W[\tau_1, \tau_2; x^i(\tau), \theta(\tau)] \\
&= \int_{\tau_1}^{\tau_2} \left\{ \left[\dot{\theta}\frac{\partial\lambda}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{\theta}\frac{\partial\lambda}{\partial y^i}\right)\right]\delta x^i - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{\theta}\frac{\partial\lambda}{\partial\dot{\theta}} + \lambda\right)\delta\theta\right\}\mathrm{d}\tau \\
&\quad + \left[\dot{\theta}\frac{\partial\lambda}{\partial y^i}\bar{\delta}x^i - \dot{\theta}\left(\frac{\partial\lambda}{\partial y^j}y^j + \dot{\theta}\frac{\partial\lambda}{\partial\dot{\theta}}\right)\delta\tau\right]_{\tau_1}^{\tau_2}
\end{aligned} \tag{14}$$

where

$$\lambda(x^i, y^j, \dot{\theta}) = \mathcal{L}\left(x^i, \frac{y^j}{\dot{\theta}}\right) \tag{15}$$

is obviously a homogeneous function of degree zero in the variables $(y^j, \dot{\theta})$. Thus, due to the Euler identity for homogeneous functions, the sum of the last two terms in the last square bracket in (14) must vanish

$$\frac{\partial\lambda}{\partial y^j}y^j + \dot{\theta}\frac{\partial\lambda}{\partial\dot{\theta}} = 0. \tag{16}$$

Moreover, due to (13), the variation $\bar{\delta}x^i(\tau)$ vanishes at the integration boundaries and, as a result, the variation principle $\bar{\delta}W = 0$ leads to the $n + 1$ equations of motion

$$\dot{\theta}\frac{\partial\lambda}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{\theta}\frac{\partial\lambda}{\partial y^i}\right) = 0 \tag{17}$$

$$-\left(\dot{\theta}\frac{\partial\lambda}{\partial\dot{\theta}} + \lambda\right) = C \tag{18}$$

(where C is a fixed constant of integration) on the functions $x^i(\tau)$ and $\theta(\tau)$. With the aid of (15) and (16), equation (18) can be written in the form

$$E\left(x^i, \frac{y^j}{\dot{\theta}}\right) = C \tag{19}$$

where

$$E(q^i, v^j) = \frac{\partial \mathcal{L}}{\partial v^k}(q^i, v^j)v^k - \mathcal{L}(q^i, v^j). \tag{20}$$

Equation (19), in accordance with the implicit-function theorem, determines $\dot{\theta}$ as a function $\phi(x^i, y^j)$ of the variables x^i and y^j, and, furthermore, it permits one to express the partial derivatives of ϕ in terms of those of the function E. From the last property, one easily finds that (see appendix (A2))

$$\frac{\partial \phi}{\partial y^k}(x^i, y^j)y^k = \phi(x^i, y^j) \tag{21}$$

which means that the function ϕ determined by (19) is necessarily homogeneous of degree one in the variables y^j. One should note that (19) determines θ as a function of the variables x^i and y^j if, and only if,

$$\frac{\partial E}{\partial \dot{\theta}}(x^i, y^j, \dot{\theta}) \neq 0$$

which implies, with the aid of (20), that†

$$\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j}(x^k, y^l)y^i y^j \neq 0.$$

If (19) is solved for $\dot{\theta}$, and the function $\phi(x^i, y^j)$ is known, one can substitute it for $\dot{\theta}$ in (17) and obtain n differential equations for the functions $x^i(\tau)$ which renders a parametric description of a trajectory corresponding to the motion determined by the functions $q^i(t)$. The point, which we will now demonstrate, is that the equations resulting from such a substitution are again Euler–Lagrange equations, but with a certain new Lagrangian $L_J(x^i, y^j)$ whose form will also be found.

When making the substitution $\dot{\theta} = \phi(x^i, y^j)$, one must first compute all the derivatives which appear in (17), to be able, in the next step, to substitute into these equations the expressions for $\dot{\theta}$ and $\ddot{\theta}$. This procedure can be facilitated by the following observations.

First, if $\dot{\theta} = \phi(x^i, y^j)$, then

$$\ddot{\theta} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\phi(x^i, y^j)) = \frac{\partial \phi}{\partial x^k}\dot{x}^k + \frac{\partial \phi}{\partial y^k}\dot{y}^k \tag{22}$$

whence, if for any function $F(x^i, y^j, \dot{\theta})$ we introduce the notation

$$\tilde{F}(x^i, y^j) = F(x^i, y^j, \phi(x^k, y^l))$$

† This inequality might be considered as a condition on velocities for which the Jacobi procedure is applicable. For instance, action (2) cannot be reduced to (1) if the velocities are null vectors with respect to g_{ij}.

then

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}(F(x^i, y^j, \dot\theta))\right]_{\dot\theta=\phi(x^k,y^l)} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\tilde F(x^k, y^l)). \tag{23}$$

Next, making use of (15) and (19), we have

$$\begin{aligned}
\left[\dot\theta\frac{\partial\lambda}{\partial x^i}\right]_{\dot\theta=\phi(x^k,y^l)} &= \phi(x^k, y^l)\left[\frac{\partial\tilde\lambda}{\partial x^i} - \frac{\partial\phi}{\partial x^i}\left(\frac{\partial\lambda}{\partial\dot\theta}\right)_{\dot\theta=\phi(x^k,y^l)}\right] \\
&= \phi(x^k, y^l)\frac{\partial\tilde\lambda}{\partial x^i} + \frac{y^j}{\phi(x^k, y^l)}\frac{\partial\phi}{\partial x^i}\left[\frac{\partial\mathcal{L}}{\partial v^j}\left(x^r, \frac{y^s}{\dot\theta}\right)\right]_{\dot\theta=\phi(x^k,y^l)} \\
&= \phi(x^k, y^l)\frac{\partial\tilde\lambda}{\partial x^i} + (\tilde\lambda(x^k, y^l) + \tilde E(x^k, y^l))\frac{\partial\phi}{\partial x^i} \\
&= \frac{\partial}{\partial x^i}[(\tilde\lambda(x^k, y^l) + C)\phi(x^k, y^l)]
\end{aligned} \tag{24}$$

and similarly

$$\begin{aligned}
\left[\dot\theta\frac{\partial\lambda}{\partial y^i}\right]_{\dot\theta=\phi(x^k,y^l)} &= \phi(x^k, y^l)\left[\frac{\partial\tilde\lambda}{\partial y^i} - \frac{\partial\phi}{\partial y^i}\left(\frac{\partial\lambda}{\partial\dot\theta}\right)_{\dot\theta=\phi(x^k,y^l)}\right] \\
&= \phi(x^k, y^l)\frac{\partial\tilde\lambda}{\partial y^i} + \frac{y^j}{\phi(x^k, y^l)}\frac{\partial\phi}{\partial y^i}\left[\frac{\partial\mathcal{L}}{\partial v^j}\left(x^r, \frac{y^s}{\dot\theta}\right)\right]_{\dot\theta=\phi(x^k,y^l)} \\
&= \phi(x^k, y^l)\frac{\partial\tilde\lambda}{\partial y^i} + (\tilde\lambda(x^k, y^l) + \tilde E(x^k, y^l))\frac{\partial\phi}{\partial y^i} \\
&= \frac{\partial}{\partial y^i}[(\tilde\lambda(x^k, y^l) + C)\phi(x^k, y^l)].
\end{aligned} \tag{25}$$

Thus, as a consequence of the substitution of $\dot\theta = \phi(x^k, y^l)$ into (17), we obtain the equations

$$\frac{\partial}{\partial x^i}[(\tilde\lambda(x^k, y^l) + C)\phi(x^k, y^l)] - \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{\partial}{\partial y^i}[(\tilde\lambda(x^k, y^l) + C)\phi(x^k, y^l)]\right\} = 0 \tag{26}$$

where

$$\tilde\lambda(x^k, y^l) = \mathcal{L}\left(x^k, \frac{y^l}{\phi(x^i, y^j)}\right)$$

which are again Euler–Lagrange equations, but with the Jacobi Lagrangian

$$\begin{aligned}
L_J(x^k, y^l) &= (\tilde\lambda(x^k, y^l) + C)\phi(x^k, y^l) \\
&= y^j\left[\frac{\partial\mathcal{L}}{\partial v^j}\left(x^r, \frac{y^s}{\dot\theta}\right)\right]_{\dot\theta=\phi(x^k,y^l)}
\end{aligned} \tag{27}$$

which additionally depends on the values of the integration constant C. This constant also enters the definition of ϕ by means of (19). As a result, (27) really determines a one-parametric family of Lagrangians.

Whereas the second equality in (27) can be found in the literature (e.g. in [1]), we would like to indicate that the first equality in (27) seems to be more convenient for applications. Furthermore, the first equality in (27) resembles the field-theoretical Fokkerian action described in [3]; cf also [4].

Let us observe that, due to the fact that $\phi(x^i, y^j)$ is homogeneous of degree one in y^j, then the function $\tilde{\lambda}(x^i, y^j)$ must be homogeneous in y^j of degree zero and the Lagrangian $L_J(x^i, y^j)$ is again homogeneous in y^j of degree one. In the appendix, it is shown, that if the Lagrangian $\mathcal{L}$ is non-singular, i.e. if

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j}\right) \neq 0$$

then the rank of the matrix $(\partial^2 L_J / \partial y^i \partial y^j)$ is $n-1$. This means that L_J is then a Lagrangian of the type considered in the Finslerian geometry [8].

From the procedure presented here, it follows that finding a solution that determines a motion of the dynamical system described by the original action (3) can now be reduced to first solving the Euler–Lagrange equations of the Jacobi Lagrangian (27), which define the path of the motion in terms of a set of functions $x^i(\tau)$, and secondly by determining a relation, given by a function $t = f(\tau)$, between the parameter τ (used for the description of the path) and the Newtonian time t from the energy condition (19), which now takes the form of a differential equation for the function f:

$$E\left(x^i(\tau), \frac{\dot{x}^j(\tau)}{\dot{f}(\tau)}\right) = C. \tag{28}$$

3. The solution of the inverse Jacobi problem

As we have seen, a Jacobi Lagrangian L_J is always a homogeneous Lagrangian of the type considered in the Finslerian geometry. A natural question arises as to whether any Lagrangian which is of the Finslerian type can be considered as a Jacobi Lagrangian for a dynamical system based on a non-singular Lagrangian $\mathcal{L}$. As was shown in [8], the answer to that question is positive, since any Lagrangian L of the Finslerian type is a Jacobi Lagrangian for the L^2, which is non-singular. This solution cannot, however, be a unique one, since, as follows from section 2, any non-singular conservative Lagrangian $\mathcal{L}$ (not necessarily homogeneous of degree two in velocities) leads to a Jacobi Lagrangian, which is homogeneous of degree one in velocities. In this section, we will present a systematic method that permits us to construct a 'primary' Lagrangian $\mathcal{L}(q^i, v^j)$ for which an *a priori* given Lagrangian $L(x^i, y^j)$, homogeneous of degree one in y^j, is its Jacobi Lagrangian. Simultaneously, the method will reveal which additional data must be given to make the solution of the problem unique.

The Euler–Lagrange equations for a homogeneous Lagrangian L

$$\frac{\delta L}{\delta x^i} = 0 \tag{29}$$

if taken alone, do not admit a unique solution. In Finsler geometry [8], in order to obtain a unique solution, these equations are completed by an algebraic equation

$L(x^i(\tau), y^j(\tau)) = \text{constant}$, which introduces an affine parametrization. Instead of doing this, one can, however, supplement (29) by any other algebraic equation of the form

$$\tilde{E}(x^i, y^j) = C \tag{30}$$

where $\tilde{E}$ is a given function such that

$$\det \begin{vmatrix} \dfrac{\partial^2 L}{\partial y^\alpha \partial y^i} \\ \dfrac{\partial \tilde{E}}{\partial y^j} \end{vmatrix} \neq 0 \tag{31}$$

where $\alpha = 1, \ldots, n-1$. The adding of (30) to (29) leads again to a unique solution which describes the same world line as before, but now, in general, in terms of a non-affine parameter.

Now we would like to find such a Lagrangian $\mathcal{L}(q^i, v^j)$ for which a given homogeneous Lagrangian $L(x^i, y^j)$ would be its Jacobi Lagrangian, whereas (30) would be a consequence (cf (28)) of the energy conservation law for $\mathcal{L}(q^i, v^j)$.

The solution to this problem will be found in two steps. In the first step, a family of Lagrangians $\mathcal{L}$ will be constructed for which a given function $E(q^i, v^j)$ is its energy function. Since a Lagrangian and its energy function are related to each other by a Legendre transformation (20), we can consider the equation

$$v^1 \frac{\partial \mathcal{L}}{\partial v^1} + \cdots + v^n \frac{\partial \mathcal{L}}{\partial v^n} - \mathcal{L} = E \tag{32}$$

as a partial differential equation for an unknown function $\mathcal{L}(q^i, v^j)$. In this equation, v^i are independent variables and q^j are treated as parameters. By suppressing the dependence on the parameters, (32) can be treated as a linear inhomogeneous partial differential equation of the first order for $\mathcal{L}(v^i)$. In accordance with the general method of solving such equations, $\mathcal{L}$ will be sought in the form of an implicit function

$$V\left(\mathcal{L}, v^1, \ldots, v^n\right) = 0. \tag{33}$$

Therefore, (32) will become a linear homogeneous differential equation

$$v^1 \frac{\partial V}{\partial v^1} + \cdots + v^n \frac{\partial V}{\partial v^n} + (\mathcal{L} + E) \frac{\partial V}{\partial \mathcal{L}} = 0 \tag{34}$$

for the function $V(\mathcal{L}, v^1, \ldots, v^n)$. The corresponding characteristic equations

$$\frac{\mathrm{d}v^1}{v^1} = \cdots = \frac{\mathrm{d}v^n}{v^n} = \frac{\mathrm{d}\mathcal{L}}{\mathcal{L} + E} = \mathrm{d}\tau \tag{35}$$

can then easily be solved explicitly in the form

$$\begin{aligned} v^i(\tau) &= c_i \mathrm{e}^\tau \\ \mathcal{L}(\tau) &= \mathrm{e}^\tau \left[\int E(v^i(\tau)) \mathrm{e}^{-\tau}\, \mathrm{d}\tau + c \right] \end{aligned} \tag{36}$$

where τ now parametrizes the solutions and where the indefinite integral is used, with c_i and c being $n+1$ integration constants.

Solving (34) requires knowledge of a complete set of independent first integrals of (35). We can construct them by eliminating τ from the solutions (36). Thus, a possible complete set of the first integrals of (35) is given by the set of functions

$$\psi_i(v^j) = \frac{v^i}{\sqrt{v^k v^k}} = \tilde{c}_i \tag{37}$$

$$\psi_0(v^j, \mathcal{L}) = \frac{\mathcal{L}}{\sqrt{v^k v^k}} - \int E(v^j(\tau)) \frac{1}{\sqrt{v^k(\tau) v^k(\tau)}} \, \mathrm{d}\tau = \tilde{c} \tag{38}$$

where the summation convention applies to all indices ranging from 1 to n and the relations

$$\mathrm{e}^\tau = \frac{\sqrt{v^i v^i}}{\sqrt{c_i c_i}} \qquad \tilde{c}_i = \frac{c_i}{\sqrt{c_k c_k}} \qquad \tilde{c} = \frac{c}{\sqrt{c_k c_k}} \tag{39}$$

have been used. The indefinite integral in (38) can be simplified after the integration variable τ is replaced by $\rho = (v^i v^i)^{1/2}$. We then have

$$\mathrm{d}\rho = \frac{v^i \, \mathrm{d}v^i}{\sqrt{v^k v^k}} = \frac{v^i \, \mathrm{d}(c_i \mathrm{e}^\tau)}{\sqrt{v^k v^k}} = \rho \, \mathrm{d}\tau \tag{40}$$

and

$$I = \int E(v^i(\tau)) \frac{1}{\sqrt{v^k(\tau) v^k(\tau)}} \, \mathrm{d}\tau = \left(\int E(\tilde{c}_i \rho) \frac{1}{\rho^2} \, \mathrm{d}\rho \right)_{\text{subst}} \tag{41}$$

where the index 'subst' means that first the integration should be performed, and then afterwards the expressions

$$\rho = \sqrt{v^k v^k} \qquad \tilde{c}_i = \frac{v^i}{\sqrt{v^k v^k}} \tag{42}$$

should be substituted for the variables ρ and $\tilde{c}_i$. Thus, in (41) we first have to compute the integral

$$\int E(\tilde{c}_1 \rho, \ldots, \tilde{c}_n \rho) \frac{1}{\rho^2} \, \mathrm{d}\rho \tag{43}$$

and we may, meanwhile, disregard the way in which the integration variable was defined. To facilitate the computation, we introduce a new variable

$$\kappa = \frac{\sqrt{v^k v^k}}{\rho}$$

where, from the point of view of the integration in (43), $\sqrt{v^k v^k}$ is just a constant factor. Thus,

$$\int E(\tilde{c}_1 \rho, \ldots, \tilde{c}_n \rho) \frac{1}{\rho^2} \, \mathrm{d}\rho = -\frac{1}{\sqrt{v^k v^k}} \int E\left(\frac{v^i}{\kappa}\right) \mathrm{d}\kappa. \tag{44}$$

Performing now the substitution (42), we obtain the first integral (38) in the form

$$\psi_0(v^i, \mathcal{L}) = \frac{1}{\sqrt{v^k v^k}} \left\{ \mathcal{L} + \left(\int E\left(\frac{v^i}{\kappa}\right) \mathrm{d}\kappa \right)_{\kappa=1} \right\}. \tag{45}$$

In accordance with the general method of solving first-order partial differential equations, the general solution of (34) is of the form

$$V = V(\psi_0(v^i, \mathcal{L}), \psi_1(v^i), \ldots, \psi_n(v^i)) \tag{46}$$

where V is an arbitrary function in $n+1$ variables. Therefore, the general solution of (32) is given in an implicit form as

$$V\left(\frac{1}{\sqrt{v^k v^k}} \left\{ \mathcal{L} + \left(\int E\left(\frac{v^i}{\kappa}\right) \mathrm{d}\kappa \right)_{\kappa=1} \right\}, \frac{v^1}{\sqrt{v^k v^k}}, \ldots, \frac{v^n}{\sqrt{v^k v^k}} \right) = 0 \tag{47}$$

or

$$\mathcal{L} = -\left(\int E\left(\frac{v^i}{\kappa}\right) \mathrm{d}\kappa \right)_{\kappa=1} + \sqrt{v^k v^k}\, \Phi\left(\frac{v^1}{\sqrt{v^k v^k}}, \ldots, \frac{v^n}{\sqrt{v^k v^k}} \right) \tag{48}$$

where Φ is another arbitrary function. In other words, a Lagrangian $\mathcal{L}(q^i, v^j)$ determined by a given energy function $E(q^i, v^j)$ is

$$\mathcal{L}(q^i, v^j) = -\left(\int E\left(q^i, \frac{v^i}{\kappa}\right) \mathrm{d}\kappa \right)_{\kappa=1} + \Lambda(q^i, v^j) \tag{49}$$

where $\Lambda(q^i, v^j)$ is an arbitrary function, homogeneous of degree one in the variables v^j. This is a general formula that determines a class of Lagrangians $\mathcal{L}$ describing a conservative dynamical system in terms of an *a priori* assigned energy function E of the system and an arbitrary homogeneous Lagrangian Λ. Formula (49) would certainly be helpful when trying to find a solution to the inverse Lagrange problem, i.e. finding a Lagrangian that leads to a system of differential equations describing a conservative dynamical system.

Now, in the next step of the procedure, we will remove the arbitrariness of Λ by making use of the requirement that a given homogeneous Lagrangian $L(x^i, y^j)$ must be the Jacobi Lagrangian corresponding to the Lagrangian (49).

In accordance with the Jacobi procedure described in section 2, to determine the Jacobi Lagrangian, one must first solve (19) in order to express $\dot{\theta}$ as a function ϕ of the variables x^i and y^j. After the function ϕ is known, the Jacobi Lagrangian can be found from (27). The application of this formula requires replacing in the Lagrangian $\mathcal{L}(q^i, v^j)$ the variables q^i by x^i and v^j by $y^j/\phi(x^k, y^l)$. Since Λ in (49) is a homogeneous function, we have

$$\mathcal{L}\left(x^i, \frac{y^j}{\phi(x^k, y^l)}\right) = -\left[\int E\left(x^i, \frac{y^j}{\kappa\phi(x^k, y^l)}\right) \mathrm{d}\kappa \right]_{\kappa=1} + \frac{1}{\phi(x^k, y^l)} \Lambda(x^i, y^j). \tag{50}$$

Making use of the first equality in (27), we obtain

$$L_J(x^i, y^j) = \left\{ -\left[\int E\left(x^i, \frac{y^j}{\kappa\phi(x^k, y^l)}\right) \mathrm{d}\kappa \right]_{\kappa=1} + \frac{1}{\phi(x^k, y^l)} \Lambda(x^i, y^j) + C \right\} \phi(x^i, y^j). \tag{51}$$

If we now equate the above expression and the known homogeneous Lagrangian $L(x^i, y^j)$, we obtain an equation which permits us to determine the unknown function Λ as

$$\Lambda(x^i, y^j) = L(x^i, y^j) + \left[\int E\left(x^i, \frac{y^j}{\kappa\phi(x^k, y^l)}\right) \mathrm{d}\kappa \right]_{\kappa=1} \phi(x^i, y^j) - C\phi(x^i, y^j). \tag{52}$$

Throughout this equation, we can change the variables (x^i, y^j) to (q^i, v^j) and substitute the expression for Λ into (49). We obtain

$$\mathcal{L}(q^i, v^j) = \left\{ \int \left[E\left(q^i, \frac{v^j}{\kappa\phi(q^k, v^l)}\right) - E\left(q^i, \frac{v^j}{\kappa}\right) \right] \mathrm{d}\kappa \right\}_{\kappa=1} + L(q^i, v^j) - C\phi(q^i, v^j). \tag{53}$$

By introducing a new integration variable $\kappa\phi(q^i, v^j)$ in the first of the integrals in (53), we can convert the two indefinite integrals into a definite one, obtaining

$$\mathcal{L}(q^i, v^j) = L(q^i, v^j) - C\phi(q^i, v^j) + \int_1^{\phi(q^k, v^l)} E\left(q^i, \frac{v^j}{\kappa}\right) \mathrm{d}\kappa. \tag{54}$$

Equation (54) represents the final formula which determines the Lagrangian $\mathcal{L}(q^i, v^j)$ in terms of an *a priori* given energy function $E(q^i, v^j)$ and an *a priori* given homogeneous Lagrangian $L(x^i, y^j)$ which, for a selected value of the energy constant C, is the Jacobi Lagrangian for $\mathcal{L}(q^i, v^j)$. The energy function defines, by means of (19), the function $\phi(x^i, y^j)$ which is also required for defining the explicit expression for $\mathcal{L}(q^i, v^j)$.

4. Examples

To illustrate the procedure decribed in section 2, let us take a Lagrangian of the form

$$\mathcal{L}(q^i, v^j) = T(q^i, v^j) + I(q^i, v^j) - V(q^i) \tag{55}$$

where T is the 'kinetic' energy, which is assumed to be a homogeneous function of degree two in the velocities v^j

$$T(q^i, kv^j) = k^2 T(q^i, v^j). \tag{56}$$

The function I is the 'interaction' energy, being homogeneous of degree one in the velocities

$$I(q^i, kv^j) = kI(q^i, v^j) \tag{57}$$

and V is the 'potential' energy.

The assumptions (56) and (57) imply the following Euler identities

$$\frac{\partial T}{\partial v^k} v^k = 2T \qquad \frac{\partial I}{\partial v^k} v^k = I. \tag{58}$$

Taking these identities into account, one can easily show that, in the present case, the energy function (20) is

$$E(q^i, v^j) = T(q^i, v^j) + V(q^i). \tag{59}$$

The construction of the Jacobi Lagrangian, corresponding to (55), starts by solving (19) with respect to $\dot{\theta}$. In our example, (19) takes the form

$$T\left(x^i, \frac{y^j}{\dot{\theta}}\right) + V(q^i) = C \tag{60}$$

and, due to homogeneity assumptions, its solution is

$$\dot{\theta} = \sqrt{\frac{T(x^i, y^j)}{C - V(x^l)}}. \tag{61}$$

The function ϕ introduced in section 2 is now given by the right-hand side of (61). It is evident that, in this example, ϕ is homogeneous of degree one in the variables y^j, which is in accordance with the general property (21). The next element required for the construction of the Jacobi Lagrangian (27) is the function $\tilde{\lambda}$, which is defined as

$$\tilde{\lambda}(x^i, y^j) = \mathcal{L}\left(x^i, \frac{y^j}{\phi(x^k, y^l)}\right). \tag{62}$$

In the present case, it is

$$\tilde{\lambda}(x^i, y^j) = \frac{T(x^i, y^j)}{\phi^2(x^k, y^l)} + \frac{I(x^i, y^j)}{\phi(x^k, y^l)} - V(x^i). \tag{63}$$

Substituting the expressions (61) for ϕ and (63) for $\tilde{\lambda}$ into (27), we find the final form of the Jacobi Lagrangian corresponding to (55):

$$L_J(x^i, y^j) = 2T(x^i, y^j)\sqrt{C - V(x^k)} + I(x^i, y^j). \tag{64}$$

In the particular case of

$$T(q^i, v^j) = \tfrac{1}{2}g_{ij}(q^k)v^i v^j \qquad I(q^i, v^j) = \frac{e}{c}A_i(q^k)v^i \qquad V(q^i) = 0 \qquad C = \tfrac{1}{2}$$

we obtain

$$L_J(x^i, y^j) = \sqrt{g_{ij}(x^k)y^i y^j} + \frac{e}{c}A_i(x^k)y^i$$

which, in particular, demonstrates how action (1) may be deduced from (2).

As another example, formula (49) may be used to find all the Lagrangians $\mathcal{L} = \mathcal{L}(q^i, v^j)$ which admit an energy function of the form

$$E(q^i, v^j) = T(q^i, v^j) + V(q^i)$$

where T is again a homogeneous function of degree two in v^j. Since

$$\int \left[T\left(q^i, \frac{v^j}{\kappa}\right) + V(q^i)\right] \mathrm{d}\kappa = T(q^i, v^j)\int \frac{\mathrm{d}\kappa}{\kappa^2} + V(q^i)\int \mathrm{d}\kappa$$

then from (49) we have

$$\mathcal{L}(q^i, v^j) = T(q^i, v^j) - V(q^i) + \Lambda(q^i, v^j)$$

where Λ is an arbitrary function, homogeneous of degree one in the variables v^j.

Yet another application of the present formalism to the standard homogeneous dynamics can be found in [8].

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Appendix

In accordance with the implicit-function theorem, by virtue of (19), we can calculate the derivatives of the function ϕ with respect to the variables x^i and y^j:

$$\frac{\partial \phi}{\partial x^i} = \phi^2 \frac{\partial E}{\partial q^i} \left(\frac{\partial E}{\partial v^k} y^k \right)^{-1} \tag{A1}$$

$$\frac{\partial \phi}{\partial y^j} = \phi \frac{\partial E}{\partial v^j} \left(\frac{\partial E}{\partial v^k} y^k \right)^{-1} . \tag{A2}$$

Starting from definition (27) of the homogeneous Lagrangian L_J, with the aid of (A2) and (20), we obtain

$$\frac{\partial^2 L_J}{\partial y^i \partial y^j} = \left(\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \frac{\partial^2 \mathcal{L}}{\partial v^m \partial v^n} y^m y^n - \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^m} y^m \frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^n} y^n \right) \left(\phi \frac{\partial^2 \mathcal{L}}{\partial v^m \partial v^n} y^m y^n \right)^{-1} . \tag{A3}$$

Let

$$r = \operatorname{rank} \left(\frac{\partial^2 L_J}{\partial y^i \partial y^j} \right)$$

where $r < n$. To prove that $r = n - 1$, as stated in the discussion following (27), suppose that $r < n - 1$. It can easily be checked using (A3) that, always,

$$\frac{\partial^2 L_J}{\partial y^i \partial y^j} y^j = 0 \tag{A4}$$

i.e. the vector $(y^1, \ldots, y^n)$ is an eigenvector of the matrix $(\partial^2 L_J / \partial y^i \partial y^j)$ belonging to an eigenvalue equal to zero. Because $r < n - 1$, there must exist another vector, say $(z^1, \ldots, z^n)$, which is linearly independent of $(y^1, \ldots, y^n)$, such that

$$\frac{\partial^2 L_J}{\partial y^i \partial y^j} z^j = 0. \tag{A5}$$

Contracting (A3) with z^j, using (A5), gives, after some manipulations,

$$\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \left(\left(\frac{z^j}{\frac{\partial^2 \mathcal{L}}{\partial v^m \partial v^n} y^m z^n} \right) - \left(\frac{y^j}{\frac{\partial^2 \mathcal{L}}{\partial v^m \partial v^n} y^m y^n} \right) \right) = 0. \tag{A6}$$

Because the vectors y^i and z^j are linearly independent, the vector in the brackets in (A6) is a non-zero vector. Thus, (A6) implies that zero is the eigenvalue of the matrix $(\partial^2 \mathcal{L} / \partial v^i \partial v^j)$, which means that $\det(\partial^2 \mathcal{L} / \partial v^i \partial v^j) = 0$. As a result, if $\det(\partial^2 \mathcal{L} / \partial v^i \partial v^j) \neq 0$, we must always have $r = n - 1$.

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